ELSEVIER Journal of Geometry and Physics 19 (1996) 379–398

JOURNAL OF GEOMETRY AND PHYSICS

Yang–Mills fields with nonquadratic energy

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Received 3 May 1995

Abstract

A gauge-invariant nonlinear Hodge–de Rham system is introduced. These equations have the same relation to the Yang–Mills equations that the conventional nonlinear Hodge equations have to the equations of classical Hodge theory. Conditions are given under which weak solutions are locally Hölder continuous. The existence of solutions is proven for variational points of a certain class of nonquadratic energy functionals.

PACS: 02.30.Jr, 02.40.Vh
Keywords: Yang–Mills equations; Nonlinear Hodge equations; p-harmonic maps
1991 MSC: 58E15, 81T13

1. Introduction and statement of the results

There are many well-known examples of variational problems on manifolds, but so far research into variational problems on fiber bundles seems to have been restricted mainly to the study of the Yang–Mills equations and other variational points of quadratic energy functionals. There is a natural way to construct a class of variational points of nonquadratic functionals which includes the Yang–Mills equations—as well as other interesting equations—as special cases. This is to extend the nonlinear Hodge equations, originally defined for Riemannian manifolds, to vector bundles.

The *nonlinear Hodge equations* [12,14] are systems of the form

$$\mathrm{d}\omega = 0, \qquad \delta(\rho(Q)\omega) = 0. \tag{1.1}$$

Here $\omega \in \Gamma(M, \Lambda^p(T^*M))$ is a section of the pth exterior power of the cotangent bundle of a compact n-dimensional Riemannian manifold M; $\mathrm{d} : \Lambda^p \to \Lambda^{p+1}$ is the flat exterior derivative on p-forms; $\delta : \Lambda^p \to \Lambda^{p-1}$ is the formal adjoint of d_j:

SSDI 0393-0440(95)00038-0

$$Q = Q(\omega) = *(\omega \wedge *\omega) = \langle \omega, \omega \rangle;$$

$* : \Lambda^p \to \Lambda^{n-p}$ is the Hodge involution; $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded C^1-function satisfying

$$0 < \rho^2(Q) + 2\rho(Q)\rho'(Q)Q < \infty \tag{1.2}$$

for $Q < Q_C \leq \infty$.

The equations

$$\delta(\rho(Q)\omega) = 0$$

are the Euler–Lagrange equations for the energy functional

$$E_M(\omega) = \int\limits_M \int\limits_0^Q \rho(t)\, \mathrm{d}t\, \mathrm{d}M,$$

where $\mathrm{d}M$ is the volume form on M. Thus solutions of (1.1) are critical points of $E_M(\omega)$ with respect to an appropriate cohomology class of admissible forms.

Eqs. (1.1) generalize the linear equations of classical Hodge theory

$$\mathrm{d}\omega = 0, \qquad \delta\omega = 0. \tag{1.3}$$

For this reason Eqs. (1.1) are said to yield a *nonlinear Hodge theory.*

Here we consider the effect on the theory of requiring critical points of $E_M(\omega)$ to possess certain symmetries which can be represented as invariance under the action of a compact Lie group G. In this case we imagine that ω takes values in the Lie algebra associated with the structure group of a vector bundle X over M. The fiber of the adjoint bundle $Ad(X)$ is the Lie algebra $\mathfrak{G}$ of G. The fiber of the automorphism bundle $Aut(X)$ is the group G itself. If A is a connection on X with curvature F_A, then A has a local representation as a $\mathfrak{G}$-valued 1-form with curvature

$$F_A = \mathrm{d}A + \tfrac{1}{2}[A, A],$$

where [,] is the Lie bracket on $\mathfrak{G}$-valued forms. These objects will be invariant under the action of sections $g \in \Gamma(Aut(X))$, called *gauge transformations*. Such maps act on the Lie algebra by conjugation. If G is compact, then they can be taken to be unitary. In fact, we assume that $G \subset SO(L)$. Then the trace inner product metric on $SO(L)$ induces a metric on G.

Under this construction critical points of $E_M(F_A)$ will satisfy the system

$$D_A F_A = 0, \qquad D_A^*(\rho(Q)F_A) = 0, \tag{1.4}$$

where $D_A = \mathrm{d} + [A,]$ is the exterior covariant derivative with formal adjoint D_A^* and $Q = \langle F_A, F_A \rangle$. Eqs. (1.4) yield a *doubly* nonlinear Hodge theory, in which in addition to the nonlinearity in F_A represented by the coefficient $\rho(Q)$, the flat derivatives d and δ of (1.1) have been replaced by bundle operators D_A and D_A^* which depend on the connection A.

[The first system in Eqs. (1.4) is the Bianchi identity for curvature 2-forms; the second is the variational equations for $E_M(F_A)$.]

Eqs. (1.4) have the same relation to the Yang–Mills equations that Eqs. (1.1) have to the Hodge equations (1.3), since if $\rho(Q) \equiv 1$, Eqs. (1.4) degenerate to the Yang–Mills equations

$$D_A^* F_A = 0.$$

Given the presence of gauge freedom and the absence of decomposition theorems analogous to those for d and δ, the existence of weak solutions of (1.4) is dependent on the particular choice of ρ; this choice determines the admissible class of finite-energy connections (see Corollary 1.2). Thus the natural question to ask is whether weak solutions of (1.4) are Hölder continuous.

The word *regular* has two distinct meanings in nonlinear Hodge theory. A Hölder continuous weak solution is of course said to be regular. In addition, the function $\rho(Q)$ in Eqs. (1.1) is said to be regular if the derivative of $Q\rho^2(Q)$ is positive and bounded; this is the content of (1.2) and in this case (1.1) is a uniformly elliptic system. By contrast, the function $\rho(Q)$ is said to be *admissible* (another unfortunate choice of terminology) if condition (1.2) holds for $Q < Q_C$.

In order to solve (1.1) for admissible ρ, one first solves (1.1) for regular ρ and shows that a smooth, unique solution exists which depends continuously on the data; in the case of nonlinear Hodge theory the data is a prescribed cohomology class. A now standard argument due to Shiffman [10] then implies that a solution to (1.1) with admissible data also exists. (See the appendix to [11] for a simple proof of Shiffman's result.)

This regularization argument does not extend in any obvious way to the bundle case. There is no natural cohomology class for F_A. Although dA is of course a closed 2-form, there is no reason to believe that a connection 1-form A satisfying Eqs. (1.4) is an extremal point of $E_M(F_A)$ with respect to a cohomology class of closed 2-forms. Thus there does not seem to be any way to show that solutions depend smoothly on the cohomology data in the sense required for regularization as in [11]. An alternative would be to show that solutions depend continuously on boundary data. However, the boundary-value problem for the nonlinear Hodge equations is unknown even for solutions of (1.1) unless ω is a 0-form. Marini has shown the existence and regularity of solutions to the Dirichlet and Neumann problems for the Yang–Mills equations over a compact Riemannian manifold of dimension 4 [5]. This argument does not extend in any obvious way to solutions of (1.4). Marini's proof strongly uses the fact that the Yang–Mills equations in a good gauge are an elliptic system with diagonal principal part. There is thus an important decoupling that occurs in the Yang–Mills case that does not occur in our case.

However, we *can* solve the regular problem, which involves absorbing the extra nonlinearities that do not appear in (1.1). We therefore let $Q = |F_A|^2 + m^2$, where m is the pointwise norm of a constant nonzero 2-density. For clarity we shall generally ignore lower-order terms that arise as a result of this construction and write simply $Q = |F_A|^2 > 0$. As the important geometry of the problem occurs in the bundle, we take M to be flat. This condition can be weakened, by applying for example arguments analogous to those in [11] or in Section 7 of [2]. In Sections 3 and 4 we prove:

Theorem 1.1. *Let M be a bounded open type-A domain of $\mathbb{R}^n$. Let F_A be a weak solution to Eqs.* (1.4) *on a bundle X over M for $\rho(Q(F_A))$ satisfying*

$$0 < \kappa \leq \rho^2(Q) + 2\rho(Q)\rho'(Q)Q \leq N < \infty \tag{1.5}$$

for constants κ and N. Suppose that $F_A \in L^P(M)$ for some $P > \frac{1}{2}n$. Then F_A is equivalent via a continuous gauge transformation to a Hölder continuous solution of (1.4).

The definition of type-A domains is reviewed in Section 2. The L^P and Hölder spaces for F_A are defined in the standard ways:

If $v \in \Gamma(M, Ad(X) \otimes \Lambda^r(T^*M))$, then

$$\|v\|_{L^P(M)} = \left(\int_M \{*Tr(v \wedge *v)\}^{P/2}\, \mathrm{d}M \right)^{1/P} = \left(\int_M \langle v, v\rangle^{P/2}\, \mathrm{d}M \right)^{1/P},$$

where $\langle\ ,\ \rangle$ denotes the inner product on $Ad(X) \otimes \mathfrak{C}$ induced by the metrics on M and G.

We say that $F_A \in C^{0,\gamma}(M)$ if each of its components are.

Well-known arguments (e.g., [9]) similarly yield Sobolev spaces of Lie algebra-valued sections.

The proof of Theorem 1.1 would not change essentially if the connection A were replaced by an arbitrary p-form Ξ on M for which

$$F_\Xi = \mathrm{d}\Xi + [A, \Xi] \quad \text{and} \quad D_A F_\Xi = 0.$$

In fact this would result in a considerably easier problem, since Eqs. (1.4) would no longer be strongly coupled in D_A and D_A^*: these operators would be linear in Ξ.

Henneaux and Teitelboim [3] have developed a gauge theory for electromagnetism in which the gauge potential A is replaced by an arbitrary p-form. In order for the system to be located in space–time the gauge group must be $U(1)$, an abelian group. In this case the Lie bracket vanishes and $F_A = \mathrm{d}A$. In this system the Yang–Mills equations reduce to the Hodge equations and our system (1.4) would reduce to (1.1). If in (1.1) we choose $p = 0$ and

$$\rho(Q) = \left(1 - \frac{\gamma - 1}{2}|\omega|^2\right)^{1/(\gamma-1)} \tag{1.6}$$

for $\gamma > 1$, then the system is identical to the equations for the stationary polytropic flow ω of a compressible fluid [12]. If $|\omega| < \frac{1}{2}(\gamma - 1)$, then the flow is said to be *subsonic*.

Using different arguments from ours, Luckhaus [4] has studied the higher regularity of minimizers of the L^P norm of the gradient of a map between Riemannian manifolds (see also [2] and the references therein). As we remarked in connection with the regularization question, a minimizing hypothesis is not appropriate to solutions of (1.4) due to the absence of well-defined boundary-value problems and the fact that $\rho(Q)$ does not in general define a Hilbert space of finite-energy connections.

For similar reasons we impose an L^P hypothesis on the solution instead of defining an admissible class of finite-energy connections. However, finite-energy solutions will satisfy the L^P hypothesis if the function ρ is chosen appropriately. For example, let

$$\rho(Q) = (|m|^2 + |F_A|^2)^{\alpha-1}, \tag{1.7}$$

where $\alpha > \frac{1}{4}n$. Then

$$E_M(F_A) = E_M^{\alpha}(F_A) = \frac{1}{\alpha}\int_M (|m|^2 + |F_A|^2)^{\alpha} * 1. \tag{1.8}$$

This functional is Palais–Smale, so the existence of weak solutions to (1.4) in this case follows from standard variational theory [8,9]. Moreover, $\rho^2(Q)+2\rho(Q)\rho'(Q)Q$ is positive for all values of Q provided $\alpha > \frac{1}{2}$. Finally, the admissible class of finite-energy connections lies in a defined Sobolev space.

Fix an arbitrary smooth base connection D_0. Then a connection is *admissible* with respect to (1.8) [i.e., has finite energy (1.8)] if it is an element of the set of connections

$$\{D = D_0 + A \mid A \in H^{1,2\alpha}(M, Ad(X) \otimes T^*M)\}.$$

Since $F_A = \mathrm{d}A + A \wedge A$, the curvature F_A of an admissible connection A satisfies $F_A \in L^P$ for $P > \frac{1}{2}n$.

The functional described in (1.8) will satisfy the conditions of Theorem 1.1 provided we restrict α to the interval $(\frac{1}{4}n, 1]$, which implies that $2 \leq n < 4$. In this case we have a local existence theorem.

Corollary 1.2. *Let M be defined as in Theorem* 1.1 *for $n < 4$. Then the functional $E_{\mathrm{M}}^{\alpha}(F_A)$ defined by* (1.8) *possesses Hölder continuous critical points whenever $\alpha \in (\frac{1}{4}n, 1]$.*

The functional of Eq. (1.8) has a potential physical interpretation as the (nonquadratic) energy of a pure Yang–Mills field. A qualitatively similar model for higher dimensions already exists in the physics literature; see [17] and the references therein. Notice that although the existence theorem (Corollary 1.2) does not apply to the functional (1.8) when $n \geq 4$ or $\alpha > 1$, the regularity theorem (Theorem 1.1) does apply.

If $n \geq 4$, then a hypothesis of finite Yang–Mills energy ($E_M(F_A) < \infty$ for $\rho \equiv 1$) is not sufficient to guarantee the condition $F_A \in L^P$ for $P > \frac{1}{2}n$. Some regularity can be shown under the assumption that the Yang–Mills energy is small outside of a prescribed singular set or that $F_A \in L^P$ for $P = \frac{1}{2}n$ (see [15] for dimension 4 and [7] and the references therein for higher dimensions). It is not clear how applicable such arguments would be to solutions of Eqs. (1.4).

Essentially for notational convenience, we restrict our attention in the subsequent arguments to the case $n > 2$. Thus we avoid considering the special case of the two-dimensional Sobolev Theorem, for which the relevant inequalities actually become stronger.

2. Background of the proof

The idea of the proof of Theorem 1.1 is to show that solutions of (1.1) and (1.4) are close when considered as points in an appropriate space. We then apply the following regularity result for solutions of (1.1).

Theorem 2.1 (Sibner [11], Section 3, Remark 3 and Section 4, Theorem 4.1). *Let ω be a weak solution of* (1.1) *on M with ρ satisfying* (1.5). *Then $\omega \in C^{0,\mu}(M)$ for some $\mu > 0$.*

In comparing systems (1.1) and (1.4), we use a mean-value formula which is essentially Lemma 1.1 of [11]:

Lemma 2.2 (cf. Sibner [11], Lemma 1.1). *Let*

$$J(x,v) = \rho(Q(v))v(x).$$

Then under the assumptions of Theorem 1.1

$$J(\xi,\mu) - J(\eta,\nu) = H(\mu - \nu) + K(\xi - \eta),$$

where H is a positive-definite matrix with finite entries and

$$|K| \leq C(|\mu(x)| + |\nu(x)|).$$

The proof of Lemma 1.1 of [11] is identical to the proof for Lie algebra-valued sections.

The L^P assumption of Theorem 1.1 allows us to apply standard results on the existence of good gauges.

Theorem 2.3 (Uhlenbeck [16], Theorem 1.3). *Let $X = B^n \times \mathbb{R}^L, G \subset SO(L), 2P \geq n$, $D = \mathrm{d} + \tilde{A}$* for *$\tilde{A} \in H^{1,P}(B^n, \mathbb{R}^L \times \mathfrak{G})$. Then $\exists\, \kappa(n) > 0$ and $c(n) < \infty$ such that if*

$$\|F\|^{n/2}_{L^{n/2}(B)} \leq \kappa(n),$$

then D is gauge equivalent by an element $s \in H^{2,P}(B^n, G)$ to a connection $\mathrm{d} + A$, *where A satisfies*

$$d^*A = 0, \qquad \|A\|_{H^{1,P}} \leq c(N)\|F\|_{L^P}.$$

The gauge whose existence is guaranteed by Theorem 2.3 is called a *Hodge gauge*.

Another useful gauge is the *exponential gauge*. In the unit n-disc $B_1(0)$ centered at the origin of coordinates in $\mathbb{R}^n$ this gauge is defined by the conditions $A_r = 0$ and $A = 0$ at the center of the ball. (Here r denotes the radial component.) In such a gauge we have a pointwise estimate

$$|A(x)| \leq \frac{|x|}{2} \sup_{|y| \leq |x|} |F_A(y)|. \tag{2.1}$$

See [15] for details.

If $F_A \in L^P$ for $P > \frac{1}{2}n$, then the gauge transformation s in Theorem 2.3 is continuous by the Sobolev Theorem and the gauge fixing preserves the topology of X. Thus we can fix a gauge and understand the notion of a weak solution of Eqs. (1.4) in the usual sense.

The proof given in the next sections relies on two well-known results from elliptic theory. We review them here.

A bounded domain Ω of $\mathbb{R}^n$ is said to be of *type A* if $\forall x \in \Omega$ and $R \in (0, diam\ \Omega)$ there exists a constant $A > 0$ such that $diam(\Omega \cap B_R(x)) \geq AR^n$. As an example, any Lipschitz domain satisfies this requirement.

Definition 2.4. Let Ω be a bounded connected open set in $\mathbb{R}^n$. A function $f \in L^P(\Omega)$, $P \geq 1$ is said to be an element of the *Campanato space* $\mathcal{L}^{P,l}$ if $l \geq 0$ and

$$\sup_{x_0 \in \Omega,\, 0<r<\operatorname{diam}\Omega} r^{-l} \int_{\Omega \cap B_r(x)} |f - (f)_{r,x_0}|^P * 1 < \infty.$$

Here

$$(f)_{r,x} = \frac{1}{|B_r(x)|} \int_{B_r(x)} f * 1.$$

The space $\mathcal{L}^{P,l}$ is a Banach space under the norm

$$\|f\|_{\mathcal{L}^{P,l}(\Omega)} = \|f\|_{L^P(\Omega)} + \left\{ \sup_{x \in \Omega,\, 0<r<\operatorname{diam}(\Omega)} r^{-l} \int_{B_r(x)} |f(x) - (f)_{r,x}|^P * 1 \right\}^{1/P},$$

where $P \geq 1, l \geq 0$.

For the properties of these spaces see Chapter 3 of [1]. However, the following property is crucial.

Theorem 2.5 (Campanato [1], Theorem III.1.2). *If Ω is a type-A domain and $n < l \leq n + P$, then the space $\mathcal{L}^{P,l}(\Omega)$ is isomorphic to the space $C^{0,\gamma}(\Omega)$ with $\gamma = (l - n)/P$.*

Denote by $T(\Omega)$ the topological space of real functions on Ω and consider the product of m copies of $T(\Omega)$ in the natural topology. Clearly the notion of Campanato spaces and the conclusion of Theorem 2.5 extend to this space of vector-valued functions. In fact, one can by standard arguments extend these ideas to spaces of Lie algebra-valued sections with components in an appropriate L^P space.

In order to estimate the Hölder norm of F_A it will first be necessary to show that F_A is bounded. This will eventually follow from a classical result due to Morrey.

Theorem 2.6 (Morrey [6], Theorem 5.3.1). *Let $u \in H^{1,2}(\Omega)$, $u \geq 0$, and define $v = u^\lambda$ for $\lambda \in [1, 2)$. Suppose that v satisfies*

$$\int_\Omega \left\{ \sum_{ij=1}^n a^{ij} \frac{\partial v}{\partial x^i} \frac{\partial \zeta}{\partial x^j} + \zeta \left(\sum_{j=1}^n b^j \frac{\partial v}{\partial x^j} - f v \right) \right\} * 1 \leq 0$$

$\forall \zeta \in C_0^\infty(\Omega), \zeta \geq 0$, *where* a^{ij} *are bounded measurable functions,* $b_j \in L^s(\Omega)$ *for some* $S > n$ *and* $f \in L^P(\Omega)$ *for some* $P > \frac{1}{2}n$. *Also assume that the matrix* a^{ij} *satisfies for some positive constants* ν_1, ν_2 *the ellipticity condition*

$$\nu_1 |\xi|^2 \leq a^{ij}\xi_i\xi_j \leq \nu_2 |\xi|^2.$$

Then for any $x \in B_\rho(x_0) \Subset \Omega$ *we have*

$$|u(x)|^2 \leq Ca^{-n} \int\limits_{B_{\rho+a}(x_0)} u^2(y)\, \mathrm{d}^n y.$$

3. An L^∞ bound on curvature

In this and the next section we prove Theorem 1.1.

We perform the initial regularity estimates in B_2, where B_R is the n-disc of radius R centered at the origin of coordinates in $\mathbb{R}^n$. We assume that on B_2 the fiber X_x of X satisfies $X_x \simeq \mathbb{R}^L$. We denote by C generic positive constants the values of which may change from line to line; x always denotes an n-vector.

The first step is to prove that $|F_A|$ is bounded. Our method is to show that we can apply Theorem 2.6.

Let F_A be a classical solution of (1.4). We show that the scalar $u(x) = Q(x)$ satisfies a subelliptic inequality.

Let ρ satisfy the hypotheses of Theorem 1.1. Letting $Q = |F_A|^2$ and writing $\rho(F_A)$ as a function of Q we have [cf. [13], Eq. (1.2)]

$$\begin{aligned} &-\tfrac{1}{2}\Delta\langle\rho(Q)F_A, \rho(Q)F_A\rangle \\ &\quad = -2\langle\Delta(\rho(Q)F_A), \rho(Q)F_A\rangle + |\nabla(\rho(Q)F_A)|^2, \end{aligned} \tag{3.1}$$

where Δ denotes the flat Laplacian on differential forms. Since F_A satisfies (1.4) we have

$$\begin{aligned} &-\langle\Delta(\rho(Q)F_A), \rho(Q)F_A\rangle \\ &\quad = -\langle(\delta\mathrm{d} + \mathrm{d}\delta)(\rho(Q)F_A), \rho(Q)F_A\rangle \\ &\quad = -\langle\delta\mathrm{d}(\rho(Q)F_A), \rho(Q)F_A\rangle + \langle\mathrm{d} * [A, *(\rho(Q)F_A)], \rho(Q)F_A\rangle. \end{aligned} \tag{3.2}$$

Substituting (3.2) into (3.1) yields

$$\begin{aligned} &-\tfrac{1}{2}\Delta(\rho^2(Q)Q) + 2\langle\delta\mathrm{d}(\rho(Q)F_A), \rho(Q)\rangle \\ &\quad = |\nabla(\rho(Q)F_A)|^2 + \langle\mathrm{d} * [A, *(\rho(Q)F_A)], \rho(Q)F_A\rangle. \end{aligned} \tag{3.3}$$

The principal part of (3.3),

$$L(Q) = -\tfrac{1}{2}\Delta\langle\rho(Q)F_A, \rho(Q)F_A\rangle + 2\langle\delta\mathrm{d}(\rho(Q)F_A), \rho(Q)F_A\rangle,$$

has the property that $-L(Q)$ is a uniformly elliptic operator on Q. (For a proof of this statement see [13, Proposition 1.1].) We can neglect the nonnegative term and write (3.3) as the inequality

$$L(Q) \geq -C(|A|\langle\nabla(\rho(Q)F_A), \rho(Q)F_A\rangle + |\nabla A|Q\rho^2(Q)).$$

But

$$\langle\nabla(\rho(Q)F_A), \rho(Q)F_A\rangle \leq |\rho'(Q)\cdot(\nabla Q)Q\rho(Q) + \tfrac{1}{2}\rho^2(Q)\cdot\nabla Q| \\ \leq |\tfrac{1}{2}(\rho^2(Q) + 2\rho(Q)\rho'(Q)Q)\nabla Q| \leq \tfrac{1}{2}N|\nabla Q|.$$

We can write condition (1.5) in the form

$$0 < \frac{\mathrm{d}}{\mathrm{d}Q}(Q\cdot\rho^2(Q)) \leq N. \tag{3.4a}$$

Integrating (3.4a) over Q yields, for a possibly different value of N,

$$\rho^2(Q) \leq N. \tag{3.4b}$$

Thus we obtain an inequality for Q of the form

$$L(Q) \geq -C(N,\rho)\{|A||\nabla Q| + |\nabla A|Q\}. \tag{3.5}$$

Writing

$$-L(Q) = a^{ij}\frac{\partial Q}{\partial x^i}\frac{\partial Q}{\partial x^j},$$

where

$$\nu_1|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq \nu_2|\xi|^2,$$

we can write (3.5) in the weak form

$$\int_B a^{ij}\frac{\partial Q}{\partial x^i}\frac{\partial \zeta}{\partial x^j} + C\left\{|A|\left(\sum_i \frac{\partial Q}{\partial x^i}\right)\zeta + |\nabla A|Q\zeta\right\}\,\mathrm{d}^n x \leq 0. \tag{3.6}$$

Here $\zeta \in C_0^\infty(B), \zeta \geq 0$.

Lemma 3.1. *Under the hypotheses of Theorem* 1.1, $F_A \in H^{1,2}(B_{3/2})$ *provided* $\|A\|_{L^n(B_2)}$ *and* $\|F_A\|_{L^{n/2}(B_2)}$ *are sufficiently small.*

Proof. The idea of the proof is to use difference quotients in order to obtain an integral estimate on the gradient of F_A.

Use Theorem 2.3 to make a continuous gauge transformation in B_2 to a Hodge gauge.

Writing Eqs. (1.4) in a weak form, we obtain for any admissible test function ψ,

$$\int_{B_2} \langle \rho(Q(x))F(x), \mathrm{d}\psi(x)\rangle \, \mathrm{d}^n x$$
$$= -\int_{B_2} \langle *[A(x), *\rho(Q(x))F(x)], \psi(x)\rangle \, \mathrm{d}^n x. \tag{3.7}$$

But if $\psi(x)$ is admissible, then so is $\psi(x-he_i)$, where e_i is the ith basis vector, $i = 1, \ldots, n$, and $h > 0$. Thus

$$\int_{B_2} \langle \rho(Q(x))F(x), \mathrm{d}\psi(x - he_i)\rangle \, \mathrm{d}^n x$$
$$= -\int_{B_2} \langle *[A(x), *\rho(Q(x))F(x)], \psi(x - he_i)\rangle \, \mathrm{d}^n x.$$

Let $y = x - he_i$; we have

$$\int_{B_2} \langle \rho(Q(y + he_i))F(y + he_i), \mathrm{d}\psi(y)\rangle \, \mathrm{d}^n y$$
$$= -\int_{B_2} \langle *[A(y + he_i), *\rho(Q(y + he_i))F(y + he_i)], \psi(y)\rangle \, \mathrm{d}^n y. \tag{3.8}$$

Subtracting (3.7) from (3.8) (expressing both integrals in terms of the dummy variable x) and dividing both sides by h yields

$$\frac{1}{h}\int_{B_2} \langle \{\rho(Q(x + he_i))F(x + he_i) - \rho(Q(x))F(x)\}, \mathrm{d}\psi(x)\rangle \, \mathrm{d}^n x$$
$$= -\frac{1}{h}\int_{B_2} \langle *[A(x + he_i), *\rho(Q(x + he_i))F(x + he_i)], \psi(x)\rangle \, \mathrm{d}^n x$$
$$+ \int_{B_2} \langle *[A(x), *\rho(Q(x))F(x)], \psi(x)\rangle \, \mathrm{d}^n x. \tag{3.9}$$

The right-hand side of (3.9) is equal to

$$-\int_{B_2} \langle *[\Delta_{i,h}A, *\rho(Q(x + he_i))F(x + he_i)], \psi(x)\rangle \, \mathrm{d}^n x$$
$$-\int_{B_2} \left\langle *\left[A(x), \frac{*\rho(Q(x + he_i))F(x + he_i)}{h}\right], \psi(x)\right\rangle \mathrm{d}^n x$$
$$+\int_{B_2} \left\langle *\left[A(x), \frac{*\rho(Q(x))F(x)}{h}\right], \psi(x)\right\rangle \mathrm{d}^n x, \tag{3.10}$$

where

$$\Delta_{i,h}u \equiv \frac{u(x+he_i)-u(x)}{h}.$$

Substitute (3.10) into the right-hand side of (3.9) and then apply Lemma 2.2 to the differences on each side of (3.9), taking $\xi = x+he_i$, $\eta = x$, $\mu = F(x+he_i)$, and $\nu = F(x)$. We obtain

$$\begin{aligned}
&\int_{B_2} \langle \{H \cdot \Delta_{i,h}F + K\}, \mathrm{d}\psi\rangle \,\mathrm{d}^n x \\
&\quad = -\int_{B_2} \langle [\Delta_{i,h}A, *\rho(Q(x+he_i))F(x+he_i)], \psi(x)\rangle \,\mathrm{d}^n x \\
&\qquad - \int_{B_2} \langle *[A(x), *(H\cdot \Delta_{i,h}F + K)], \psi(x)\rangle \,\mathrm{d}^n x.
\end{aligned} \tag{3.11}$$

Now

$$\begin{aligned}
\Delta_{i,h}F &= \frac{F(x+he_i)-F(x)}{h} \\
&= \frac{\mathrm{d}A(x+he_i)-\mathrm{d}A(x)}{h} + \frac{1}{h}[A(x+he_i), A(x+he_i)] - \frac{1}{h}[A(x), A(x)] \\
&= \Delta_{i,h}(\mathrm{d}A) + \frac{1}{h}[A(x+he_i) - A(x) + A(x), A(x+he_i)] - \frac{1}{h}[A(x), A(x)] \\
&= \Delta_{i,h}(\mathrm{d}A) + [\Delta_{i,h}A, A(x+he_i)] + [A(x), \Delta_{i,h}A].
\end{aligned} \tag{3.12}$$

The function $\psi = \zeta\,\Delta_{i,h}A$ is an admissible test function if $\zeta \in C_0^\infty(B)$ and $\zeta(x) = 1$ for $x \in B_{3/2}$, where $\zeta(x) \geq 0\,\forall x$. Then, using (3.12), we have from (3.11)

$$\begin{aligned}
&\int_{B_2} \langle H\cdot\{\Delta_{i,h}(\mathrm{d}A) + [\Delta_{i,h}A, A(x+he_i)] + [A(x), \Delta_{i,h}A]\}, \zeta\,\mathrm{d}(\Delta_{i,h}A)\rangle \,\mathrm{d}^n x \\
&\quad + \int_{B_2} \langle K, \zeta\,\mathrm{d}(\Delta_{i,h}A)\rangle \,\mathrm{d}^n x \\
&\qquad = -\int_{B_2} \langle [\Delta_{i,h}A, *\rho(Q(x+he_i))F(x+he_i)], \zeta\,\Delta_{i,h}A\rangle \,\mathrm{d}^n x \\
&\qquad\quad - \int_{B_2} \langle *[A(x), *(H\cdot\{\Delta_{i,h}(\mathrm{d}A) + [\Delta_{i,h}A, A(x+he_i)] \\
&\qquad\qquad + [A(x), \Delta_{i,h}A]\} + K)], \zeta\,\Delta_{i,h}A\rangle \,\mathrm{d}^n x \\
&\qquad\quad - \int_{B_2} \langle (H\cdot\{\Delta_{i,h}(\mathrm{d}A) + [\Delta_{i,h}A, A(x+he_i)] \\
&\qquad\qquad + [A(x), \Delta_{i,h}A]\} + K), (\Delta_{i,h}A)\,\mathrm{d}\zeta\rangle \,\mathrm{d}^n x.
\end{aligned} \tag{3.13}$$

Here

$$0 < \theta_1 \le H \le \theta_2 < \infty \tag{3.14}$$

and

$$|K| \le C(|F(x+he_i)| + |F(x)|). \tag{3.15}$$

Using inequalities (3.14), (3.15), (3.4b), and the fact that d commutes with $\Delta_{i,h}$, we have

$$\begin{aligned}
&\theta_1 \int_{B_2} \zeta |\Delta_{i,h}(\mathrm{d}A)|^2 \,\mathrm{d}^n x \\
&\le C(\theta_2, N, \max|\nabla\zeta|) \left\{ \int_{B_2} (|A(x+he_i)| + |A(x)|)|\Delta_{i,h}A||\Delta_{i,h}(\mathrm{d}A)| \,\mathrm{d}^n x \right. \\
&\quad + \int_{B_2} (|F(x+he_i)| + |F(x)|)|\Delta_{i,h}(\mathrm{d}A)| \,\mathrm{d}^n x \\
&\quad + \int_{B_2} |\Delta_{i,h}A|^2 |F(x+he_i)| \,\mathrm{d}^n x \\
&\quad + \int_{B_2} (|A(x)|+1)|\Delta_{i,h}(\mathrm{d}A)||\Delta_{i,h}A| \,\mathrm{d}^n x \\
&\quad + \int_{B_2} (|A(x)|+1)|\Delta_{i,h}A|^2(|A(x+he_i)| + |A(x)|) \,\mathrm{d}^n x \\
&\quad \left. + \int_{B_2} (|F(x+he_i)| + |F(x)|)|\Delta_{i,h}A| \,\mathrm{d}^n x \right\}.
\end{aligned} \tag{3.16}$$

Label the six integrals on the right-hand side of (3.16) as $I_1, I_2, \ldots, I_6$, respectively. We estimate these integrals individually.

Let ϵ be a small positive constant. Young's inequality implies that

$$\begin{aligned}
I_1 &\le C(\epsilon) \int_{B_2} (|A(x+he_i)| + |A(x)|)^2 |\Delta_{i,h}A|^2 \,\mathrm{d}^n x + \epsilon \int_{B_2} |\Delta_{i,h}(\mathrm{d}A)|^2 \,\mathrm{d}^n x \\
&\le C(\epsilon) \left(\int_{B_2} (|A(x+he_i)| + |A(x)|)^n \,\mathrm{d}^n x \right)^{2/n} \left(\int_{B_2} |\Delta_{i,h}A|^{2n/(n-2)} \,\mathrm{d}^n x \right)^{(n-2)/n} \\
&\quad + \epsilon \int_{B_2} |\Delta_{i,h}(\mathrm{d}A)|^2 \,\mathrm{d}^n x.
\end{aligned}$$

Applying the Minkowski, Young and Sobolev inequalities to the right-hand side of the estimate, we obtain

$$I_1 \le C\left(\|A(x+he_i)\|^2_{L^n(B_2)} + \|A(x)\|^2_{L^n(B_2)}\right)$$
$$\times\left(\int_{B_2} |\nabla(\Delta_{i,h}A)|^2\,\mathrm{d}^n x + \int_{B_2} |\Delta_{i,h}A|^2\,\mathrm{d}^n x\right) + \epsilon\int_{B_2} |\Delta_{i,h}(\mathrm{d}A)|^2\,\mathrm{d}^n x, \qquad (3.17)$$

using the fact that

$$\int_{B_2} |\nabla|\Delta_{i,h}A||^2\,\mathrm{d}^n x \le \int_{B_2} |\nabla(\Delta_{i,h}A)|^2\,\mathrm{d}^n x.$$

The Gaffney–Gärding inequality yields

$$\int_{B_2} |\nabla(\Delta_{i,h}A)|^2\,\mathrm{d}^n x \le C\left(\int_{B_2} |\Delta_{i,h}(\mathrm{d}A)|^2\,\mathrm{d}^n x + \int_{B_2} |\Delta_{i,h}A|^2\,\mathrm{d}^n x\right). \qquad (3.18)$$

In (3.18) we have used the fact that d and δ commute with $\Delta_{i,h}$ and $\delta A = 0$. Substituting (3.18) into (3.17) yields an estimate for I_1.

$$I_2 \le C(\epsilon)\int_{B_2} (|F(x+he_i)| + |F(x)|)^2\,\mathrm{d}^n x + \epsilon\int_{B_2} |\Delta_{i,h}(\mathrm{d}A)|^2\,\mathrm{d}^n x;$$

$$I_3 \le \left(\int_{B_2} |\Delta_{i,h}A|^{2n/(n-2)}\,\mathrm{d}^n x\right)^{(n-2)/n}\left(\int_{B_2} |F(x+he_i)|^{n/2}\,\mathrm{d}^n x\right)^{2/n}$$
$$\le C\|F(x+he_i)\|_{L^{n/2}(B_2)}\left(\int_{B_2} |\nabla|\Delta_{i,h}A||^2\,\mathrm{d}^n x + \int_{B_2} |\Delta_{i,h}A|^2\,\mathrm{d}^n x\right);$$

apply (3.18) to get an estimate similar to that obtained for I_1, but with coefficients involving the $L^{n/2}$-norm of $F(x+he_i)$ rather than the L^n-norms of $A(x+he_i)$ and $A(x)$.

$$I_4 \le C(\epsilon)\int_{B_2} (|A|+1)^2|\Delta_{i,h}A|^2\,\mathrm{d}^n x + \epsilon\int_{B_2} |\Delta_{i,h}(\mathrm{d}A)|^2\,\mathrm{d}^n x,$$

so I_4 can be estimated by essentially the same argument that was used in estimating I_1. The presence of the number 1 in the first integral on the right in the estimate of I_4 is harmless, as $A \in H^{1,2}(B_{n/2})$. The estimation of I_5 is similar to that of I_1 and I_4, since

$$I_5 \le \int_{B_2} (|A(x+he_i)| + |A(x)+1|)^2|\Delta_{i,h}A|^2\,\mathrm{d}^n x.$$

Finally,

$$I_6 \leq \left(\int\limits_{B_2} (|F(x+he_i)| + |F(x)|)^2 \, \mathrm{d}^n x \right)^{1/2} \left(\int\limits_{B_2} |\Delta_{i,h} A|^2 \, \mathrm{d}^n x \right)^{1/2}.$$

With the aid of the above estimates we absorb small terms on the left in (3.16), using the hypotheses on the L^n-norm of A and the $L^{n/2}$-norm of F on B_2. We obtain, since $F_A \in L^P(B_2)$ and $A \in H^{i,P}(B_2)$ for $P > \frac{1}{2}n$ by hypothesis and by Theorem 2.3, respectively,

$$\int\limits_{B_{3/2}} |\Delta_{i,h}(\mathrm{d}A)|^2 \, \mathrm{d}^n x \leq C < \infty.$$

Letting h tend to zero and summing over $i = 1, 2, \ldots, n$, we have $\mathrm{d}A \in H^{1,2}(B_{3/2})$. But

$$\int |\nabla F_A|^2 * 1 \leq \int |\nabla(\mathrm{d}A)|^2 * 1 + \int |\nabla(A \wedge A)|^2 * 1$$
$$\leq C \left(1 + \int |\nabla A|^2 |A|^2 * 1 \right),$$

and

$$\int |\nabla A|^2 |A|^2 * 1 \leq \|A\|^2_{L^n} \|\nabla A\|^2_{L^{2n/(n-2)}}.$$

The first norm on the right can be seen to be finite by applying Theorem 2.3 and the Sobolev Theorem. The second norm on the right is manifestly finite for $n \geq 6$. Otherwise, write

$$\|\nabla A\|^2_{L^{2n/(n-2)}} \leq C \left(\|\nabla|\nabla A|\|^2_{L^2} + \|\nabla A\|^2_{L^2} \right).$$

An argument similar to (3.18) yields

$$\left\| \nabla |\nabla A| \right\|^2_{L^2} \leq C \left(\|\mathrm{d}|\nabla A|\|^2_{L^2} + \|\nabla A\|^2_{L^2} \right).$$

Notice also that ∇ commutes with d in the above inequality (possibly taking a mollification limit of the components of A) and that

$$\int |\nabla|\mathrm{d}A||^2 * 1 \leq \int |\nabla(\mathrm{d}A)|^2 * 1 < \infty.$$

This completes the proof of Lemma 3.1. □

The hypotheses on the L^n-norm of A and the $L^{n/2}$-norm of F_A can always be satisfied locally by a conformal transformation that leaves (1.4) invariant.

We are not yet in a position to invoke Theorem 2.6 and conclude that the L^∞-norm of F_A is bounded. We could do this via Lemma 3.1 and inequality (3.6), letting $\nu = Q$, if we could take the constant λ in Theorem 2.6 to be equal to 2, but λ is required to be strictly less than 2. The following lemma is intended to overcome this problem.

Lemma 3.2. *Under the hypotheses of Theorem* 1.1, $|F_A|^\tau \in H^{1,2}(B_{5/4})$ *for some* $\tau > 1$.

Proof. We can write (3.6) in the form

$$2\int\limits_{B_{3/2}} a^{ij} u \frac{\partial u}{\partial x^i}\frac{\partial \zeta}{\partial x^j}\,\mathrm{d}^n x \le C_N \left\{ \int\limits_{B_{3/2}} |A||u||\nabla u|\zeta\,\mathrm{d}^n x + \int\limits_{B_{3/2}} |\nabla A| u^2 \zeta\,\mathrm{d}^n x \right\},$$

where $u = |F_A|$ and $\zeta \in C_0^\infty(B_{3/2}), \zeta \ge 0$. Choose

$$\zeta = (u_k + \delta)^{2\tau-2}\eta^2$$

for $\eta \in C_0^\infty(B_{3/2}), \eta \ge 0, \delta > 0, \tau > 1$. The sequence $\{u_k\}$ is chosen to be increasing and so that $\lim_{k\to\infty} u_k = u$. We have

$$\begin{aligned}
&\int\limits_{B_{3/2}} a^{ij} u (u_k+\delta)^{2\tau-3}\nabla u \cdot \nabla(u_k)\eta^2 * 1 \\
&\quad \le C_1(\nu_2,\tau)\int\limits_{B_{3/2}} u|\nabla u|(u_k+\delta)^{2\tau-2}\eta|\nabla\eta| * 1 \\
&\qquad + C_2(N,\tau)\left(\int\limits_{B_{3/2}} |A||u||\nabla u|(u_k+\delta)^{2\tau-2}\eta^2 * 1\right) \\
&\qquad + C_3(N)\left(\int\limits_{B_{3/2}} |\nabla A| u^2 (u_k+\delta)^{2\tau-2}\eta^2 * 1\right).
\end{aligned} \tag{3.19}$$

We can bound the right-hand side of (3.19) by the number

$$\begin{aligned}
&C(N,\nu_2,\tau,\eta,\nabla\eta)\left(\int\limits_{B_{3/2}} (|A|+1)(u+\delta)^{2\tau-1}|\nabla u| * 1 + \int\limits_{B_{3/2}} |\nabla A|(u+\delta)^{2\tau} * 1\right) \\
&\quad \equiv i_1 + i_2;
\end{aligned}$$

$$\begin{aligned}
i_1 &\le \||A|+1\|_{p_1}\left(\int (u+\delta)^{(2\tau-1)q_1}|\nabla u|^{q_1} * 1\right)^{1/q_1} \\
&\le \||A|+1\|_{p_1}\|u+\delta\|^{2\tau-1}_{(2\tau-1)q_1p_2}\|\nabla u\|_{q_1q_2}; \\
i_2 &\le \|\nabla A\|_{p_3}\left(\int (u+\delta)^{2\tau q_3} * 1\right)^{1/q_3}.
\end{aligned}$$

Choose $p_1 = n$ and $p_2 = 2(n-1)/(n-2)$. If $n > 6$, choose $p_3 = \frac{1}{2}n$. If $n \le 6$, choose $p_3 = 2n/(n-2)$. In each case choose q_i to be the conjugate of p_i. If $n > 6$, choose $\tau > 1$ so that $2n(2\tau - 1)/(n-2) \le P$ for $P > \frac{1}{2}n$. If $n \le 6$, choose $\tau > 1$ so that $2n(2\tau - 1)/(n-2) \le P$ for $P > 2n/(n-2)$. The resulting norms are seen to be finite under the hypotheses of Theorem 1.1 by applying the Sobolev Theorem to Lemma 3.1 and to the inequality of Theorem 2.3.

Use Fatou's Lemma to let k tend to infinity in (3.19); let δ tend to 0. Then we can replace (3.19) by the inequality

$$\nu_1 \int_{B_{3/2}} \eta^2 |\nabla u^\tau|^2 \, \mathrm{d}^n x \le C < \infty.$$

The proof of Lemma 3.2 is completed by letting $\eta = 1$ on $B_{5/4}$. □

Now apply Theorem 2.6 in $B_{5/4}$, taking $u = |F_A|^\tau$ and using (3.6). The coefficients $|A|$ and $|\nabla A|$ in (3.6) can be seen to satisfy the hypotheses of Theorem 2.6 by applying the Sobolev Theorem to the inequality of Theorem 2.3. We have proven:

Theorem 3.3. *Under the hypotheses of Theorem* 1.1, $|F_A|$ *is bounded on the unit n-disc in* M.

4. The Hölder continuity of the connection

In this section B denotes the unit n-disc centered at the origin of coordinates in $\mathbb{R}^n$. Consider a 1-form ϕ satisfying (1.1) in the sense that

$$\delta(\rho(Q(\mathrm{d}\phi))\,\mathrm{d}\phi) = 0, \tag{4.1}$$

where ρ satisfies (1.5). We can choose ϕ so that $\phi_\vartheta = A_\vartheta$ on ∂B, where the subscript ϑ denotes the tangential component. By possibly adding a 0-form h satisfying the Poisson problem

$$\delta\,\mathrm{d}h = -\delta\phi, \quad \text{in } B,$$
$$h_\vartheta = 0, \quad \text{on } \partial B,$$

we can construct a new 1-form $\tilde{\phi} = \phi + \mathrm{d}h$ satisfying (4.1) and the constraint $\delta\tilde{\phi} = 0$. In the sequel we shall assume that this has been done and suppress the tilde.

It is demonstrated in the proof of Theorem 2.1 that ϕ satisfies the inequality

$$\int_{B_r(x_0)} |\mathrm{d}\phi - (\mathrm{d}\phi)_{r,x_0}|^2 * 1 \le C r^{n+2\gamma} \tag{4.2}$$

for $\gamma \in (0,1]$ and x_0 an arbitrary interior point of M. The Hölder continuity of $\mathrm{d}\phi$ follows from (4.2) by Theorem 2.5.

We initially compare $\mathrm{d}A$ and $\mathrm{d}\phi$ in a ball B_r of radius r centered at the origin of coordinates in M. (Inequalities (1.5) and (3.4b) imply that condition (A) of [11] is satisfied by ϕ.)

Lemma 4.1. $\forall r \in (0, 1)$,

$$\int_{B_r} |\mathrm{d}A - \mathrm{d}\phi|^2 * 1 \leq C r^{2+n} \|F_A\|_{L^\infty(B_r)}.$$

Proof. Use Theorem 2.3 to make a continuous gauge transformation in B to a Hodge gauge. Eqs. (1.4) and (4.1) imply that in B

$$\int_B \langle A - \phi, \delta\{\rho(Q(F_A))F_A - \rho(Q(\mathrm{d}\phi))\,\mathrm{d}\phi\}\rangle * 1$$
$$= \int_B \langle A - \phi, *[A, *\rho(Q(F_A))F_A]\rangle * 1. \tag{4.3}$$

We use Green's Theorem in the form

$$\int_B \langle \mathrm{d}u, \nu\rangle * 1 - \int_B \langle u, \delta\nu\rangle * 1 = \int_{\partial B} u_\vartheta \wedge \nu_N, \tag{4.4}$$

where ν is a p-form, u is a $(p-1)$-form, and $\nu_N = (*\nu)_\vartheta$. Applying (4.4) to (4.3) yields, using the boundary condition for (4.1),

$$\int_B \langle \mathrm{d}(A - \phi), \rho(Q(F_A))F_A - \rho(Q(\mathrm{d}\phi))\,\mathrm{d}\phi\rangle * 1$$
$$= \int_B \langle A - \phi, *[A, *\rho(Q(F_A))F_A]\rangle * 1. \tag{4.5}$$

Now apply Lemma 2.2 to the left-hand side of Eq. (4.5), taking $\mu = F_A, \nu = \mathrm{d}\phi$, and $\xi = \eta = x$. We obtain

$$\theta_1 \int_B \langle \mathrm{d}(A - \phi), (F_A - \mathrm{d}\phi)\rangle * 1 \leq \left| \int_B \langle A - \phi, *[A, *\rho(Q(F_A))F_A]\rangle * 1 \right|,$$

or

$$\theta_1 \int_B \langle \mathrm{d}(A - \phi), \mathrm{d}(A - \phi)\rangle * 1$$
$$\leq \left| \int_B \langle A - \phi, *[A, *\rho(Q(F_A))F_A]\rangle * 1 \right| + \theta_2 \left| \int_B \langle \mathrm{d}(A - \phi), A \wedge A\rangle * 1 \right|.$$

Here θ_1 and θ_2 are the constants of Eq. (3.14).

$$\theta_1 \int_B |\mathrm{d}A - \mathrm{d}\phi|^2 * 1 \le C(\rho) \|F_A\|_{L^\infty(B)} \int_B |A - \phi||A| * 1$$
$$+ \theta_2(\|\mathrm{d}A\|_{L^\infty(B)} + \|\mathrm{d}\phi\|_{L^\infty(B)}) \int_B |A|^2 * 1. \tag{4.6}$$

But

$$\int_B |A - \phi||A| * 1 \le \epsilon^{-1} \int_B |A|^2 * 1 + \epsilon \int_B |A - \phi|^2 * 1$$
$$\le \epsilon^{-1} \int_B |A|^2 * 1 + C\epsilon |B|^{2/n} \int_B |\nabla(A - \phi)|^2 * 1$$
$$\le \epsilon^{-1} \int_B |A|^2 * 1 + C\epsilon |B|^{2/n} \int_B |\mathrm{d}A - \mathrm{d}\phi|^2 * 1. \tag{4.7}$$

In (4.7) we have used the Poincarè and Gaffney–Gärding inequalities and we have taken into account the fact that A and ϕ are d-coclosed and equal on ∂B. (Compare (4.7) with the first few lines in the proof of [16, Lemma 2.5].) Substituting (4.7) into (4.6) yields

$$\theta_1 \int_B |\mathrm{d}A - \mathrm{d}\phi|^2 * 1$$
$$\le C\|F_A\|_{L^\infty(B)} \left\{ \epsilon^{-1} \int_B |A|^2 * 1 + \epsilon |B|^{2/n} \int_B |\mathrm{d}A - \mathrm{d}\phi|^2 * 1 \right\}$$
$$+ \theta_2(\|\mathrm{d}A\|_{L^\infty(B)} + \|\mathrm{d}\phi\|_{L^\infty(B)}) \int_B |A|^2 * 1. \tag{4.8}$$

Rescaling (4.8) by the conformal transformation $x \to rx, 0 < r < 1$, yields

$$(1 - C\epsilon r^4) \int_{B_r} |\mathrm{d}A - \mathrm{d}\phi|^2 * 1 \le C \int_{B_r} |A|^2 * 1. \tag{4.9}$$

Choose an exponential gauge. Apply inequality (2.1) on B_r and integrate to obtain

$$\int_{B_r} |A|^2 * 1 \le C r^{2+n} \|F_A\|_{L^\infty(B_r)}. \tag{4.10}$$

Substituting (4.10) into (4.9) proves the lemma. □

Since the mean value $(\mathrm{d}A)_{r,0}$ of $\mathrm{d}A$ on B_r minimizes the functional

$$E_c \equiv \int_{B_r} |\mathrm{d}A - c|^2 * 1$$

over all constant 2-forms c, we have, using Lemma 4.1,

$$\int_{B_r} |\mathrm{d}A - (\mathrm{d}A)_{r,0}|^2 * 1 \leq \int_{B_r} |\mathrm{d}A - (\mathrm{d}\phi)_{r,0}|^2 * 1$$
$$\leq \int_{B_r} |\mathrm{d}A - \mathrm{d}\phi|^2 * 1 + \int_{B_r} |\mathrm{d}\phi - (\mathrm{d}\phi)_{r,0}|^2 * 1$$
$$\leq C(r^{n+2} + r^{n+2\gamma}) \leq C r^{n+2\gamma}. \tag{4.11}$$

(Alternatively, we could have replaced $(\mathrm{d}A)_{r,0}$ by a solution of a variational problem for the functional E_c. An argument along these lines is constructed in Section 4 of [11].)

We would like to repeat estimate (4.11) over domains $B_r(x_0) \in M$ such that $x_0 \neq 0$. The obstruction to doing this is the choice of gauge, since we have used estimate (2.1). In order to apply (4.11) on $B_r(x_0)$ we must show that the gauge transformations are close to the identity in the *Campanato semi-norm*:

$$\mathcal{L}^{2,0}(f; r, x_0) \equiv \|f - (f)_{r,x_0}\|_{L^2}.$$

Lemma 4.2. *If $x \in B_r(\sigma)$ and $g \in G$ is a map such that $g(A)$ satisfies* (1.4), *then*

$$\|g^{-1}(x)\,\mathrm{d}A(x)g(x) - (g^{-1}(x)\,\mathrm{d}A(x)g(x))_{r,\sigma}\|_{L^2(B_r(\sigma))} \leq C r^{\beta}$$

for some $\beta \geq 0$.

Proof. We can estimate $(g^{-1}(x)\,\mathrm{d}A(x)g(x))_{r,\sigma}$ by $g^{-1}(\sigma)(\mathrm{d}A(x))g(\sigma)$. Since g is unitary,

$$\begin{aligned}|n_1 - n_2| &\equiv \|g^{-1}(x)\,\mathrm{d}Ag(x) - g^{-1}(\sigma)\,\mathrm{d}Ag(\sigma)\|_{L_2}\\ &= \|\mathrm{d}A - g(x)g^{-1}(\sigma)\,\mathrm{d}Ag(\sigma)g^{-1}(x)\|_{L^2}\\ &= \|\mathrm{d}Ag(x)g^{-1}(\sigma) - g(x)g^{-1}(\sigma)\,\mathrm{d}A\|_{L^2}\\ &= \|\mathrm{d}A(g(x)g^{-1}(\sigma) - I) + (I - g(x)g^{-1}(\sigma))\,\mathrm{d}A\|_{L^2},\end{aligned}$$

where I is the identity transformation. Thus

$$\begin{aligned}|n_1 - n_2| &\leq 2\|\mathrm{d}A\|_{L^\infty}\|g(x)g^{-1}(\sigma) - I\|_{L^2}\\ &= 2\|\mathrm{d}A\|_{L^\infty}\|g(x) - g(\sigma)\|_{L^2}.\end{aligned}$$

But

$$\|\mathrm{d}g\|_{L^2} \leq \|A\|_{L^2} + \|gA\|_{L^2} \leq Cr,$$

where gA is the image of A under g and the constant C depends on the L^∞-norm of F (see the proof of [16, Lemma 2.4],).

We have shown that for x sufficiently close to σ, i.e., for r sufficiently small

$$\|g^{-1}(x)\,\mathrm{d}A(x)g(x) - (g^{-1}(x)\,\mathrm{d}A(x)g(x))_{r,\sigma}\|_{L^2(B_r(\sigma))} \leq Cr,$$

which proves the lemma. □

Lemma 4.2 allows us to apply the arguments leading to (4.11) on $B_r(\sigma)$, where r is sufficiently small and σ is arbitrary in M. In this way the estimates of this section extend to the interior of M by a covering argument. We conclude from Theorem 2.5 that $\mathrm{d}A$ is Hölder continuous. Thus, F_A is as well.

This completes the proof of Theorem 1.1.

Acknowledgements

The regularity arguments in Section 4 are based on a series of conversations with Professor K. Uhlenbeck about the existence of critical points for the p-Yang–Mills functional (1.8). I am also grateful to Professor R. Hardt for drawing my attention to the paper [4].

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